Orthogonal support splitting and sharp purity bounds for bosonic extensions

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Abstract

Let K be an orthogonal projector on $\mathrm{Sym}^2 \mathbb{C}^d$ with one-particle marginal equal to the identity. We give a Hilbert-orthogonal support decomposition of its compatible symmetric cubic tensors. If K admits a positive bosonic three-particle extension, its blocks have dimensions $n_i \in \{1, 3, 4, \ldots\}$, and every extension is represented by a positive semidefinite matrix C with diagonal $C_{ii} = n_i$. Consequently the minimum unnormalized purity is $\sum_i n_i^2$, the minimizing extension is unique, and a rank-one extension always exists. For $d \geq 3$, every non-basis projector admitting an extension has purity at least $d + 6$, and this bound is attained. Equality requires one three-dimensional block, singleton blocks, and zero interblock coherence; it does not specify a unique three-dimensional model. A Hesse family shows that neither a dimension-only positive spectral gap nor a fixed positive marginal-error tolerance can enforce the exact basis conclusion. The argument is self-contained; its relation to established tensor decomposition and quantum marginal results is discussed without a priority claim.

1 Setting and main conclusions

Let $V = \mathbb{C}^d$ with its Hermitian inner product. Tensor products carry the induced inner product, and $\mathrm{Sym}^m V$ denotes the subspace fixed by every permutation of the m factors. All traces are ordinary, unnormalized traces. We call K a *balanced bosonic projector* if

$$K = K^* = K^2, \qquad \operatorname{ran} K \subseteq \mathrm{Sym}^2 V, \qquad \mathrm{Tr}_2 K = I_V. \tag{1}$$

It follows that $\operatorname{rank} K = \mathrm{Tr}\, K = d$ and $\mathrm{Tr}_1 K = I_V$. We regard K as an operator on $V^{\otimes 2}$, zero outside its range. Define its compatible cubic space by

$$W(K) = (\operatorname{ran} K \otimes V) \cap \mathrm{Sym}^3 V. \tag{2}$$

An *extension* in this paper means an operator satisfying

$$R \geq 0, \qquad \operatorname{ran} R \subseteq \mathrm{Sym}^3 V, \qquad \mathrm{Tr}_3 R = K. \tag{3}$$

In particular, $\mathrm{Tr}\, R = d$. Range containment in the symmetric subspace is essential; mere invariance under permutations is a weaker condition.

We say that K has *basis form* if some orthonormal basis $(u_j)_{j=1}^d$ satisfies

$$K = \sum_{j=1}^{d} |u_j^{\otimes 2}\rangle\langle u_j^{\otimes 2}|. \tag{4}$$

The main structural results are Theorems 2.5 and 3.1. They decompose $W(K)$ into one-dimensional spaces supported on mutually orthogonal subspaces of V, then identify every extension with a weighted correlation matrix. The sharp consequence is

$$K \text{ not of basis form and (3) holds} \quad \Longrightarrow \quad \operatorname{Tr} R^2 \geq d+6 \qquad (d \geq 3). \tag{5}$$

Equality is possible in every such dimension. In dimension two, every balanced bosonic projector already has basis form, whether or not extension existence is assumed.

2 Support saturation and orthogonal splitting

For $T \in \operatorname{Sym}^3 V$, let its *one-particle support* be

$$U(T) = \operatorname{ran}\big(\operatorname{Tr}_{12}|T\rangle\langle T|\big).$$

For nonzero T, this is the smallest subspace $U \subseteq V$ with $T \in \operatorname{Sym}^3 U$. Indeed, zero expectation of the positive one-particle marginal on $U^\perp$ implies that the tensor has no component in $V^{\otimes 2} \otimes U^\perp$; symmetry gives the same assertion in each slot.

We use the complex *linear dual* V^* to define the contraction map

$$A_T : V^* \longrightarrow \operatorname{Sym}^2 V, \qquad A_T(\varphi) = (I \otimes I \otimes \varphi)T. \tag{6}$$

Both $\varphi \mapsto A_T(\varphi)$ and $T \mapsto A_T$ are complex linear. In particular, no antilinear identification of V with V^* is used in the pencil argument below. The Schmidt rank across the first two slots and the last slot gives

$$\operatorname{rank} A_T = \dim U(T), \qquad \operatorname{ran} A_T \subseteq \operatorname{Sym}^2 U(T). \tag{7}$$

Equivalently, in an orthonormal basis the matrix of A_T is a flattening of the tensor coefficients, and its rank equals the rank of the corresponding one-particle reduced operator. The second inclusion follows from $T \in \operatorname{Sym}^3 U(T)$.

Lemma 2.1 (Positivity and support)**.** *If $X \geq 0$ on a finite-dimensional Hilbert space and E is an orthogonal projector with $\operatorname{Tr}(EX) = 0$, then $EX = XE = 0$. If $X \geq 0$ on $V^{\otimes 2}$ has bosonic range and $\operatorname{Tr}_2 X$ is supported on $M \subseteq V$, then $\operatorname{ran} X \subseteq \operatorname{Sym}^2 M$.*

Proof. The positive operator $X^{1/2} E X^{1/2}$ has trace zero and is therefore zero. Thus $EX^{1/2} = 0$, proving the first assertion. For the second, apply the first assertion to $E = P_{M^\perp} \otimes I$. This places the range in $M \otimes V$. Bosonic symmetry also places it in $V \otimes M$, whose intersection with the first space is $M \otimes M$. □

Lemma 2.2 (Support saturation)**.** *Suppose (1) holds and $0 \neq T \in W(K)$. Put $U = U(T)$ and $k = \dim U$, and let L be the orthogonal projector onto $\operatorname{ran} A_T$. Then*

$$K = L + K', \qquad \operatorname{Tr}_2 L = I_U, \qquad \operatorname{Tr}_2 K' = I_{U^\perp}, \tag{8}$$

where I_U and $I_{U^\perp}$ denote projectors on V. The operators L and K' are orthogonal projectors with ranges in $\operatorname{Sym}^2 U$ and $\operatorname{Sym}^2 U^\perp$, respectively. Moreover,

$$W(K) = W(L) \oplus W(K'), \tag{9}$$

where the two spaces on the right are formed on U and $U^\perp$.

Proof. Every contraction of T lies in $\operatorname{ran} K$, so $L \le K$. By (7), $\operatorname{rank} L = k$ and its range lies in $\operatorname{Sym}^2 U$. Hence

$$0 \le \operatorname{Tr}_2 L \le I_V, \qquad \operatorname{ran}(\operatorname{Tr}_2 L) \subseteq U, \qquad \operatorname{Tr}(\operatorname{Tr}_2 L) = k.$$

A positive contraction on a k-dimensional space has trace k only if all its eigenvalues equal one. Thus $\operatorname{Tr}_2 L = I_U$. Since $L \le K$, the difference $K' = K - L$ is an orthogonal projector, and its marginal is $I_{U^\perp}$. Lemma 2.1 puts its range in $\operatorname{Sym}^2 U^\perp$.

To verify (9), use a basis adapted to $U \oplus U^\perp$. A compatible cubic has no coefficient whose first two indices lie in different subspaces, since $\operatorname{ran} K$ has no such component. Symmetry moves any pair of indices to those two positions. Thus every coefficient involving both subspaces vanishes. The cubic is a sum of a tensor in $\operatorname{Sym}^3 U$ and a tensor in $\operatorname{Sym}^3 U^\perp$, and their contractions must lie in the corresponding ranges of L and K'. The reverse inclusion is immediate. □

Lemma 2.3 (A singular pencil member)**.** *If* $\dim W(K) \ge 2$*, some nonzero* $T \in W(K)$ *has* $\dim U(T) < d$*.*

Proof. Choose linearly independent $S, T \in W(K)$. The contraction maps take values in $\operatorname{ran} K$, whose dimension is d, so bases of V^* and $\operatorname{ran} K$ represent them by square $d \times d$ matrices. Consider the homogeneous polynomial

$$f(x, y) = \det(xA_S + yA_T).$$

If it vanishes identically, A_S is singular. Otherwise it is a homogeneous polynomial of positive degree over $\mathbb{C}$, and it has a zero in $\mathbb{CP}^1$. At a representative $(x, y) \ne (0, 0)$ of that zero, the tensor $xS + yT$ is nonzero by independence, while its contraction map is singular. Equation (7) proves the assertion. □

Lemma 2.4 (The binary case)**.** *For a balanced bosonic projector on a two-dimensional space,* $\dim W(K) \ge 2$*. Consequently it has basis form.*

Proof. Here $\dim \operatorname{Sym}^2 V = 3$ and $\operatorname{rank} K = 2$. Let $q : \operatorname{Sym}^2 V \to \operatorname{Sym}^2 V / \operatorname{ran} K$ be the quotient map. The kernel of

$$\operatorname{Sym}^3 V \longrightarrow (\operatorname{Sym}^2 V / \operatorname{ran} K) \otimes V, \qquad T \longmapsto (q \otimes I)T$$

is $W(K)$. Its domain has dimension four and its target has dimension two, so its kernel has dimension at least two. Lemma 2.3 produces a nonzero tensor with one-dimensional support. Lemma 2.2 then splits K along two orthogonal lines. On either line there is only one balanced projector, the projector onto its tensor square. This is (4). □

Theorem 2.5 (Orthogonal support decomposition)**.** *Every* K *satisfying* (1) *has an orthogonal decomposition*

$$V = V_1 \oplus \cdots \oplus V_r \oplus V_*, \qquad K = K_1 \oplus \cdots \oplus K_r \oplus K_*, \tag{10}$$

with the following properties.

1. *Each* K_i *is balanced on* V_i*, and*

$$W(K_i) = \mathbb{C}t_i, \qquad \|t_i\| = 1, \qquad U(t_i) = V_i.$$

   Its dimension $n_i = \dim V_i$ *is either one or at least three.*

2. *The possibly zero remainder* K_* *is balanced on* V_* *and* $W(K_*) = \{0\}$*. If* $V_* \ne \{0\}$*, then* $\dim V_* \ge 3$*.*

3. The compatible cubic space is

$$W(K) = \bigoplus_{i=1}^{r} \mathbb{C} t_i. \tag{11}$$

If $r = 0$, the active sum is empty. In every case $r \le d$. Equality holds precisely for basis form; if K is not of basis form, then $r \le d - 2$.

Proof. On any current balanced block, stop if its compatible cubic space is zero. Also stop if that space is a line generated by a tensor with full one-particle support. In all other cases there is a nonzero compatible tensor with proper support: use its generator when the space is a line, and Lemma 2.3 when its dimension is at least two. Lemma 2.2 then splits the block into two nonzero balanced blocks on strictly smaller spaces, and splits its cubic space accordingly. Dimension strictly decreases along each branch, so this procedure terminates. Combine all zero-cubic blocks into the single orthogonal remainder. Equation (9) shows that their sum still has zero cubic space and proves (11).

A one-dimensional balanced block has a one-dimensional cubic space. Lemma 2.4 excludes dimension two both for active terminal blocks and for a nonzero inactive remainder. This proves the dimension restrictions.

Put $n_* = \dim V_*$. Then

$$d - r = n_* + \sum_{i=1}^{r} (n_i - 1).$$

If the right side vanishes, all blocks are singleton blocks and the remainder is absent, giving basis form. Conversely, for (4) the compatible cubics are exactly the span of $u_j^{\otimes 3}$, since the slice condition and symmetry exclude all mixed indices. Hence $r = d$. If basis form fails, either some $n_i \ge 3$, contributing at least two to $d - r$, or $n_* \ge 3$. This gives $r \le d - 2$. □

3 Classification of all positive extensions

The preceding decomposition does not assume extension existence. The following theorem states exactly what this additional condition requires.

Theorem 3.1 (Weighted matrix classification)**.** *Let (10) be the decomposition of Theorem 2.5. An extension (3) exists if and only if $V_* = \{0\}$ and*

$$\mathrm{Tr}_3 |t_i\rangle\langle t_i| = \frac{K_i}{n_i} \qquad (1 \le i \le r). \tag{12}$$

When these conditions hold, all extensions are exactly

$$R(C) = \sum_{i,j=1}^{r} C_{ij} |t_i\rangle\langle t_j|, \qquad C \ge 0, \qquad C_{ii} = n_i. \tag{13}$$

In particular, if $D = \mathrm{diag}(n_1, \dots, n_r)$, then $C = D^{1/2} Z D^{1/2}$ for an arbitrary positive semidefinite correlation matrix Z with $Z_{ii} = 1$.

Proof. For an extension, let $P = K \otimes I$. Its marginal gives

$$\mathrm{Tr}((I - P)R) = \mathrm{Tr}((I - K)K) = 0.$$

Lemma 2.1 forces $\operatorname{ran} R \subseteq \operatorname{ran} P$. Bosonic support now implies $\operatorname{ran} R \subseteq W(K)$. Since (t_i) is an orthonormal basis of this space, R has the unique matrix representation in (13), with positivity equivalent to $C \geq 0$.

For $i \neq j$, the orthogonality $V_i \perp V_j$ in the traced slot gives

$$\operatorname{Tr}_3 |t_i\rangle\langle t_j| = 0. \tag{14}$$

For example, evaluate the partial trace in an orthonormal basis adapted to all the V_i and V_*. In each summand, at least one of the two contractions is zero. Thus the marginal equation becomes the orthogonal block equality

$$\sum_i C_{ii} \operatorname{Tr}_3 |t_i\rangle\langle t_i| = \sum_i K_i + K_*. \tag{15}$$

Its left side has no component on $\operatorname{Sym}^2 V_*$. Hence $K_* = 0$, and balance implies $V_* = \{0\}$. On the ith block, taking the trace gives $C_{ii} = \operatorname{Tr} K_i = n_i$, because $\|t_i\| = 1$. The same block equation then gives (12).

Conversely, if the remainder is absent and (12) holds, every positive C with the stated diagonal defines a positive bosonic $R(C)$. Equations (14) and (12) give its marginal $\sum_i K_i = K$. Such a matrix always exists, for example $C = D$. □

Corollary 3.2 (Exact purity and pure extensions)**.** *Suppose an extension exists. Define*

$$R_{\min} = \sum_{i=1}^{r} n_i |t_i\rangle\langle t_i|, \qquad p_{\min} = \sum_{i=1}^{r} n_i^2.$$

Every extension satisfies the exact identities

$$\operatorname{Tr} R(C)^2 = p_{\min} + \sum_{i \neq j} |C_{ij}|^2, \qquad \|R(C) - R_{\min}\|_{\mathrm{HS}}^2 = \operatorname{Tr} R(C)^2 - p_{\min}. \tag{16}$$

Thus $R_{\min}$ is the unique minimum-purity extension. Rank-one extensions always exist, and all of them are

$$R = |\psi\rangle\langle\psi|, \qquad \psi = \sum_{i=1}^{r} \sqrt{n_i}\, e^{\mathrm{i}\theta_i} t_i, \qquad \theta_i \in \mathbb{R}. \tag{17}$$

Their purity is d^2. In fact the attainable purities comprise the full interval $[p_{\min}, d^2]$.

Proof. The isometry sending the ith standard basis vector to t_i identifies $R(C)$ with C, so its squared Hilbert–Schmidt norm is $\operatorname{Tr} C^2 = \sum_{ij} |C_{ij}|^2$. Its diagonal is fixed, proving (16) and uniqueness. A positive rank-one matrix with diagonal n_i has the form vv^* with $v_i = \sqrt{n_i} e^{\mathrm{i}\theta_i}$, proving (17). Also $\|\psi\|^2 = \sum_i n_i = d$. No positive trace-$d$ operator has purity exceeding d^2. For any such v, the feasible matrices

$$C(s) = (1 - s)D + svv^*, \qquad 0 \leq s \leq 1,$$

have purity $p_{\min} + s^2(d^2 - p_{\min})$, which covers the claimed interval. □

For basis form, every $n_i = 1$ and $t_i = u_i^{\otimes 3}$ up to phases. Thus the classification specializes to ordinary correlation matrices, with minimum purity d and exact distance $\|R - R_{\min}\|_{\mathrm{HS}}^2 = \operatorname{Tr} R^2 - d$. Low purity need not make R a projector: $C = I + a(E_{12} + E_{21})$, with real a and $0 < |a| < 1$, is positive with purity $d + 2a^2$ and has eigenvalues $1 + a, 1 - a, 1, \ldots, 1$.

4 The sharp non-basis threshold and equality

Theorem 4.1 (Sharp purity threshold)**.** *Let $d \geq 3$, and suppose (1) and (3) hold. If K is not of basis form, then*

$$\operatorname{Tr} R^2 \geq d + 6. \tag{18}$$

For such a non-basis K, equality holds if and only if the support decomposition has exactly one block of dimension three and $d-3$ singleton blocks, and $R = R_{\min}$, equivalently $C_{ij} = 0$ for $i \neq j$. The bound is attained in every dimension $d \geq 3$. Therefore the strict implication

$$\operatorname{Tr} R^2 < d + 6 \quad \Longrightarrow \quad K \text{ has basis form}$$

is optimal among criteria depending only on d and an upper purity cutoff.

Proof. Extension existence removes the inactive remainder. Because basis form fails, some $n_i \geq 3$, and all other n_j are either one or at least three. Consequently

$$p_{\min} = \sum_i n_i^2 = d + \sum_i n_i(n_i - 1) \geq d + 6. \tag{19}$$

A nonzero summand on the right is at least six, with equality precisely when its block has dimension three. Hence equality in (19) requires exactly one such block and all others singleton. Equation (16) gives the additional necessary and sufficient condition of zero interblock coherence.

For attainment, choose orthonormal $e_1, \ldots, e_d$ and write

$$s_{ab} = \frac{e_a \otimes e_b + e_b \otimes e_a}{\sqrt{2}}, \qquad \tau = \frac{1}{\sqrt{6}} \sum_{\pi \in S_3} e_{\pi(1)} \otimes e_{\pi(2)} \otimes e_{\pi(3)}.$$

Set

$$K = |s_{12}\rangle\langle s_{12}| + |s_{13}\rangle\langle s_{13}| + |s_{23}\rangle\langle s_{23}| + \sum_{j=4}^{d} |e_j^{\otimes 2}\rangle\langle e_j^{\otimes 2}|,$$

$$R = 3|\tau\rangle\langle\tau| + \sum_{j=4}^{d} |e_j^{\otimes 3}\rangle\langle e_j^{\otimes 3}|.$$

The pair vectors are orthonormal. Since

$$\operatorname{Tr}_2 |s_{ab}\rangle\langle s_{ab}| = \tfrac{1}{2}(|e_a\rangle\langle e_a| + |e_b\rangle\langle e_b|), \qquad \operatorname{Tr}_3 |\tau\rangle\langle\tau| = \tfrac{1}{3}\big(|s_{12}\rangle\langle s_{12}| + |s_{13}\rangle\langle s_{13}| + |s_{23}\rangle\langle s_{23}|\big),$$

these operators satisfy (1) and (3). On the first three coordinates, compatible cubics cannot repeat an index: moving the repeated pair to the first two slots would produce a forbidden diagonal pair. Thus their space is exactly $\mathbb{C}\tau$. The singleton spaces contribute their cubes and mixed blocks contribute nothing, giving $\dim W(K) = d - 2$. Theorem 2.5 therefore excludes basis form. Finally $\operatorname{Tr} R^2 = 9 + (d-3) = d + 6$. □

The equality statement specifies block sizes and coherence, not a unique three-dimensional tensor or projector. Section 5 gives further equality examples. In particular, when $d = 3$, every extendible non-basis projector has a one-dimensional compatible cubic space and a unique extension, necessarily of purity nine. For $d = 2$, Lemma 2.4 leaves no non-basis case. For normalized states $\rho = K/d$ and $\sigma = R/d$, the sharp non-basis lower bound is $\operatorname{Tr} \sigma^2 \geq (d+6)/d^2$; for example, the unnormalized boundary is 12 when $d = 6$.

5 Hesse degeneration and the limits of exact stability

We now exhibit balanced non-basis projectors converging to a basis projector. This also distinguishes the classification above from a uniform noisy recognition theorem.

Proposition 5.1 (A three-dimensional family)**.** *In $V = \mathbb{C}^3$ with orthonormal basis e_0, e_1, e_2, let t be real and set $N_t = \sqrt{1+2t^2}$ and*

$$q_0(t) = \frac{|00\rangle + t(|12\rangle + |21\rangle)}{N_t}, \qquad q_1(t) = \frac{|11\rangle + t(|20\rangle + |02\rangle)}{N_t},$$
$$q_2(t) = \frac{|22\rangle + t(|01\rangle + |10\rangle)}{N_t}, \qquad K_t = \sum_{i=0}^{2} |q_i(t)\rangle\langle q_i(t)|.$$

Here $|ij\rangle = e_i \otimes e_j$. Each K_t is balanced. If $t \neq 0$ and $t^3 \neq 1$, then $W(K_t)$ is the line generated by

$$h_t = \frac{\sum_{i=0}^{2} |iii\rangle + t \sum_{\pi \in S_3} |\pi(012)\rangle}{\sqrt{3+6t^2}}, \tag{20}$$

and $\mathrm{Tr}_3 |h_t\rangle\langle h_t| = K_t/3$. Thus K_t is not of basis form, and its unique extension is $3|h_t\rangle\langle h_t|$, with purity nine. At $t = 0$, K_0 has basis form and $\dim W(K_0) = 3$.

Proof. The three $q_i(t)$ are unit vectors with disjoint coordinate supports, so K_t is a projector. Each q_i has a diagonal one-particle marginal, and each coordinate receives one contribution of weight $1/N_t^2$ and two of weight t^2/N_t^2. Their sum is the identity.

For a compatible cubic T, express its contraction in the third coordinate k as $\sum_i B_{ik} q_i(t)$. Symmetry of the tensor coefficients gives, when $t \neq 0$,

$$B_{00} = B_{11} = B_{22},$$

and, by comparing coefficients having two equal indices,

$$B_{01} = tB_{20} = t^2 B_{12} = t^3 B_{01},$$
$$B_{02} = tB_{10} = t^2 B_{21} = t^3 B_{02}.$$

If $t^3 \neq 1$, all off-diagonal entries vanish, so the compatible space has dimension at most one. The nonzero tensor (20) belongs to it and is normalized. Its three contractions are $q_i(t)/\sqrt{3}$, whence its marginal is $K_t/3$. The extension and non-basis claims follow from Theorems 2.5 and 3.1. At zero, the compatible cubics are precisely the three coordinate cubes. □

For $0 < t < 1/2$, the tensors h_t are not unitarily equivalent, even up to a nonzero scalar, to the triangle tensor τ. To see this, identify a symmetric tensor with its homogeneous cubic polynomial. Up to scale, (20) gives

$$f_t(x, y, z) = x^3 + y^3 + z^3 + 6txyz,$$

whereas the triangle gives xyz. A projective singular point of $f_t = 0$ would satisfy $x^2 + 2tyz = y^2 + 2tzx = z^2 + 2txy = 0$. A zero coordinate forces all three to vanish. Otherwise, multiplying the equations gives $1 = -8t^3$, impossible for positive t. Thus its projective cubic is smooth, while $xyz = 0$ is singular. An invertible linear change preserves this distinction. This gives explicit inequivalent three-dimensional equality models without asserting a classification of their unitary orbits.

Proposition 5.2 (No uniform exact stability)**.** *For a balanced K on $\mathbb{C}^d$, $d \geq 3$, let*

$$H(K) = 2I - K \otimes I - I \otimes K$$

and let $\gamma(K)$ be its smallest positive eigenvalue. No positive lower bound for $\gamma(K)$ depending only on d holds, even among projectors admitting extensions. Moreover, for every $\varepsilon > 0$ there are a non-basis balanced K and a positive bosonic R_0 with

$$\operatorname{Tr} R_0 = \operatorname{Tr} R_0^2 = d, \qquad \| \operatorname{Tr}_3 R_0 - K \|_1 < \varepsilon. \tag{21}$$

Thus no fixed positive dimension-only tolerance in the marginal, even at purity d, can force exact basis form.

Proof. Write $P = K \otimes I$ and $Q = I \otimes K$. Since $H(K) = (I - P) + (I - Q)$ is positive, its kernel is $\operatorname{ran} P \cap \operatorname{ran} Q$. A tensor in that intersection is fixed by the swaps (12) and (23), because both pair ranges are bosonic. These swaps generate all permutations, so it is fully symmetric. Conversely a symmetric tensor in $\operatorname{ran} P$ lies in $\operatorname{ran} Q$. Hence

$$\ker H(K) = W(K).$$

For K_t of Proposition 5.1, $H(K_t)$ depends continuously on t, its kernel has dimension one for $0 < |t| < 1/2$, and its kernel has dimension three at zero. Order its eigenvalues nondecreasingly. The second eigenvalue is $\gamma(K_t) > 0$ for such nonzero t, and it converges to the second eigenvalue at zero, which is zero. This proves the gap assertion in dimension three.

Now take $R_0 = \sum_{i=0}^{2} |e_i^{\otimes 3}\rangle\langle e_i^{\otimes 3}|$. It is positive and bosonic, has trace and purity three, and has marginal K_0. The differences of the three pair projectors in $K_t - K_0$ act on mutually orthogonal two-dimensional spaces. For unit vectors a, b, the trace norm of $|a\rangle\langle a| - |b\rangle\langle b|$ is $2\sqrt{1 - |\langle a, b\rangle|^2}$. Since $\langle ii, q_i(t)\rangle = 1/N_t$, it follows that

$$\|K_t - K_0\|_1 = 6\sqrt{\frac{2t^2}{1 + 2t^2}} \longrightarrow 0. \tag{22}$$

Choosing sufficiently small nonzero t proves (21) in dimension three. For larger d, adjoin $d - 3$ orthogonal singleton square projectors to K_t and their cube projectors to R_0. Trace and purity each increase by $d - 3$, and the marginal error is unchanged. The compatible-space dimensions are then $d - 2$ for nonzero t and d at zero, giving the same gap collapse by continuity. □

This obstruction concerns an *exact* conclusion under approximate input. It does not exclude estimates proving that K is close to some basis-form projector, or certificates whose constants use a rigorously established gap for the particular K. Nor does it address the separate problem of approximate idempotence and approximate balance.

6 Relation to prior work and verification scope

The direct-sum theory of homogeneous forms is established. Huang, Lu, Ye, and Zhang [1], Proposition 2.1, relate decompositions to idempotents in the commutative center. Their Lemma 3.10 identifies admissible linear combinations of gradients with center elements; Theorem 3.7 counts indecomposable summands when the center is semisimple. This is close to the contraction-space algebra used here. Their multilinear notion of orthogonality does not by itself impose the given Hermitian inner product, and general centers can have nilpotents. The explicit step in Lemma 2.2

uses projector balance to force Hilbert-orthogonal support saturation. General tensor direct-sum decomposition is not claimed as new.

Bravyi and Vyalyi [2], equations (18)–(19) in the proof of Lemma 7, use independent outer tensor coefficients to derive commutation of overlapping interaction algebras. That mechanism provides a standard route to the basis-form equality case of two-projection arguments. It is not needed for the support-splitting proof above; the full hypotheses of their Lemma 7 are not asserted here.

The symmetric-extension results of Navascués, Owari, and Plenio [3] are also relevant, but their Appendix B, Lemma 11, requires a PPT Bose symmetric extension in addition to a rank condition. Positivity in (3) does not supply that PPT hypothesis. Indeed, in the basis example after Corollary 3.2, a nonzero off-diagonal coefficient a yields, after transposition in the third factor, a 2×2 block with eigenvalues $\pm|a|$ on $|112\rangle$ and $|221\rangle$. Such extensions can have purity arbitrarily close to d.

Feng, Duan, and Ying [4], Theorem 1, characterize pure states sharing their proper reduced states through local orthogonal blocks and relative phases. This is relevant to the phase freedom in (17). The present statement instead begins with a balanced projector marginal and derives both the existence of a pure extension from any positive extension and the full positive-matrix family (13).

These comparisons locate the argument within known methods; they do not establish priority for its precise hypotheses or conclusions. A finite literature screening cannot certify that a theorem has not appeared elsewhere. Specialist review of the tensor-center and quantum-marginal connections remains appropriate.

Research and verification disclosure. The derivations, manuscript preparation, literature screening, and companion exact symbolic controls involved substantial OpenAI AI assistance. Separate AI proof audits are not independent human validation or formal proof-assistant verification. The companion controls check finite examples, contraction ranks, marginal identities, and purity normalizations; the universal statements depend on the analytic proofs above. No journal submission or publication is asserted.

References

[1] H.-L. Huang, H. Lu, Y. Ye, and C. Zhang, *On centers and direct sum decompositions of higher degree forms*, arXiv:2009.12014, version 2 (2021), Proposition 2.1, Lemma 3.10, and Theorem 3.7.

[2] S. Bravyi and M. Vyalyi, *Commutative version of the k-local Hamiltonian problem and common eigenspace problem*, Quantum Inf. Comput. **5** (2005), 187–215, arXiv:quant-ph/0308021, equations (18)–(19) in the proof of Lemma 7.

[3] M. Navascués, M. Owari, and M. B. Plenio, *The power of symmetric extensions for entanglement detection*, Phys. Rev. A **80** (2009), 052306, arXiv:0906.2731, Appendix B, Lemma 11.

[4] Y. Feng, R. Duan, and M. Ying, *Locally undetermined states, generalized Schmidt decomposition, and an application in distributed computing*, arXiv:0808.2938 (2008), Theorem 1.