A sharp positivity gap for central factorial Veneziano residues

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Abstract

We sharpen the uniform bound $D = 51/5$ in the existing finite-product manuscript to the optimal value $D_\star = (125-\sqrt{1081})/9$. After removing the polynomial at the exceptional level, all remaining central-factorial residues have nonnegative Gegenbauer coefficients exactly for $3 < D \le D_\star$. A rational certificate proves positivity of all deep trajectories on the larger interval $10 \le D \le 256/25$; the known level-five scalar then fixes the sharp endpoint. This lowers the finite-product reduction cutoff from $n = 35$ to $n = 30$, reducing the required nonexceptional finite-base residue checks from 560 to 405. This is a mathematical refinement of four-point partial-wave positivity. It establishes no higher-point or loop completion, and novelty remains provisional.

1 Statement and conventions

For integers $m \ge 0$, define

$$Q_m(y) = \prod_{r=0}^{m-1} \left(y + \frac{m-1}{2} - r \right), \qquad W_m(x) = \frac{Q_m((m+1)x/2)}{m!}, \quad Q_0 = 1. \tag{1}$$

Expand W_m in the standard Gegenbauer basis $C_j^\lambda(x)$, with real $D > 3$ and $\lambda = (D-3)/2$. The index j is a nonnegative integer spin; parity-allowed means $j \le m$ and $m - j$ even. In the type-I residue convention of Ref. [1], the physical pole label is $n = m + 1$ and $W_m = (m+1)R^{\mathrm{I}}_{m+1}$.

Theorem 1. *Every Gegenbauer coefficient of every W_m with $m \neq 2$ is nonnegative if and only if*

$$3 < D \le D_\star := \frac{125 - \sqrt{1081}}{9} = 10.2357150616058674\ldots. \tag{2}$$

At $D = D_\star$, all parity-allowed coefficients in this retained family are strictly positive except $(m, j) = (4, 0)$, which is zero. Forbidden-parity and above-degree coefficients vanish. The omitted W_2 has a negative scalar coefficient at $D_\star$.

Removing W_2 is essential: the full type-I amplitude already fails its scalar positivity test above ten. The theorem is about the retained polynomial family, not a change to the critical dimension of a physical string theory.

The upper-bound polynomial is prior work. Table 1 of Ref. [1] gives

$$B^D_{5,0} = \frac{9D^2 - 250D + 1616}{1920(D^2 - 1)}. \tag{3}$$

Our W_4 scalar is five times this expression. The candidate contribution is that this known upper bound is sufficient for every retained level and spin. The existing manuscript [3] establishes the weaker 51/5 bound and explicitly leaves this optimal-gap question open.

2 Bessel transform and recurrence

The central-factorial generating function used in Ref. [2] is

$$\sum_{m\ge 0}\frac{Q_m(y)}{m!}u^m=\frac{\exp(2y\,\operatorname{arsinh}(u/2))}{\sqrt{1+u^2/4}}. \tag{4}$$

It also follows directly by the substitutions $u=2\sinh z$, $w=e^{2z}$ in the coefficient contour, giving $\operatorname{Res}_{w=1} w^{y+(m-1)/2}/(w-1)^{m+1}=\binom{y+(m-1)/2}{m}$. For allowed parity put

$$\ell=\frac{m-j}{2},\quad c=j+2\ell+1,\quad b=j+\frac{D-1}{2},\qquad H(z)=\left(\frac{z}{\sinh z}\right)^c{}_0F_1\left(;b;\frac{c^2z^2}{4}\right)=\sum_{p\ge 0}h_pz^{2p}. \tag{5}$$

The Gegenbauer plane-wave expansion and the preceding generating function yield

$$[C_j^\lambda]W_m=\frac{c^j}{2^{m+j}(\lambda)_j}h_\ell. \tag{6}$$

Thus the sign is that of h_ℓ. Bessel representations of this type are established methods [1, 2]; the normalization here agrees with Ref. [3].

The hypergeometric factor Φ obeys $\Phi''+(2b-1)\Phi'/z-c^2\Phi=0$. Substitute $\Phi=(\sinh z/z)^cH$ and multiply by $z^2\sinh^2 z$. Coefficient comparison gives, for $p\ge 1$,

$$\Delta_p h_p=\sum_{r=2}^{p+1}\alpha_r M_{p,r}h_{p+1-r},\qquad \Delta_p=2p(2j+2p+D-3), \tag{7}$$

$$\alpha_r=\frac{2^{2r-1}}{(2r)!},\qquad q=p+1-r, \tag{8}$$

$$\begin{aligned}M_{p,r}&=c[(4\ell+4-D)r+j-2\ell+D-4]\\&\quad-2q[2q-1+2cr+D-4-4\ell].\end{aligned} \tag{9}$$

The target parameters c,b,ℓ stay fixed as p varies. All denominators Δ_p used below are strictly positive.

3 A rational certificate on an interval

We prove positivity of all targets with $\ell\ge 3$ throughout $10\le D\le 256/25$. This interval extends beyond $D_\star$; the shallow scalar, treated separately, limits the theorem.

Positive bulk. For $p\le\ell-1$, write $\ell-p=1+X$, $r=2+Y$, $p=r-1+Q$, with $X,Y,Q\ge 0$. Expansion of (9) at $D=256/25$ has twenty strictly positive rational coefficients in j,X,Y,Q. This identity is reconstructed by the accompanying code. Furthermore,

$$\frac{\partial M_{p,r}}{\partial D}=-c(r-1)-2q<0. \tag{10}$$

The bulk edges are therefore positive throughout the interval. Induction from $h_0=1$ proves $h_p>0$ for every $p\le\ell-1$.

Boundary exit weights. Set $T_0=1$ and

$$T_d=\sum_{u=0}^{d-1}T_u\alpha_{d-u+1}M_{\ell-u,d-u+1}\prod_{v=u+1}^{d-1}\Delta_{\ell-v},\qquad d=1,2. \tag{11}$$

Let $(a)_{\underline{r}} = a(a-1)\cdots(a-r+1)$. Define

$$P(j,X,Y;D) = 22500\sum_{u=0}^{2} T_u 4^{-u}(2R+2)_{\underline{2u}} M_{\ell-u,R-u+1} \prod_{v=u+1}^{2} \Delta_{\ell-v}\Big|_{R=X+3,\,\ell=X+Y+3}. \tag{12}$$

Expanding gives 182 monomials in j, X, Y. Each coefficient $a(D)$ has degree at most three. The executable certificate checks that

$$(1+U)^3 a\left(10 + \frac{6U}{25(1+U)}\right) \tag{13}$$

has strictly positive rational coefficients: 728 in total, with minimum 2500. The map $U \geq 0$ covers $[10, 256/25)$; the positive cubic coefficient gives positivity also at the upper endpoint. Consequently $P > 0$ on the entire nonnegative j, X, Y orthant and dimension interval. This finite identity controls infinitely many levels and spins.

Expanding (7) only at depths zero, one and two below h_ℓ, and stopping at first exit, gives the exact identity

$$h_\ell = \sum_{R=3}^{\ell} E_R h_{\ell-R}, \qquad E_R = \frac{\alpha_{R+1} P(j, R-3, \ell-R; D)}{22500\Delta_\ell \Delta_{\ell-1}\Delta_{\ell-2}} > 0. \tag{14}$$

This is the bounded-width first-exit method of Ref. [2], adapted as in Ref. [3]. Every landing term lies in the positive bulk. Thus $h_\ell > 0$ for all $\ell \geq 3$.

4 Shallow trajectories and global sharpness

Direct expansion at the remaining targets yields

$$h_0 = 1, \qquad h_1 = \frac{(j+3)(j+10-D)}{6(D+2j-1)}, \tag{15}$$

$$h_2 = \frac{(j+5)N(j,D)}{360(D+2j-1)(D+2j+1)}, \tag{16}$$

$$N(j,D) = 5j^3 + (153-10D)j^2 + (5D^2-192D+1570)j + 27D^2 - 750D + 4848. \tag{17}$$

For h_1, every integer $j \geq 1$ gives positivity up to $D_\star < 11$. Its $j = 0$ case is the excluded $(m, j) = (2, 0)$. For h_2 and $j \geq 1$, substitute $j = J + 1$ and use the dimension parametrization in (13), clearing degree two. The resulting polynomial in J, U has twelve strictly positive rational coefficients. For $j = 0$, $N(0, D)$ has its lower zero at $D_\star$ and is positive for $10 \leq D < D_\star$. This proves all endpoint assertions.

For $0 < \lambda \leq \mu$, the connection formula is

$$C_j^\mu(x) = \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{\lambda + j - 2r}{\lambda} \frac{(\mu)_{j-r}(\mu-\lambda)_r}{(\lambda+1)_{j-r} r!} C_{j-2r}^\lambda(x). \tag{18}$$

Every coefficient is nonnegative. Positivity at $D_\star$ therefore implies positivity at every real $3 < D < D_\star$.

To exclude *every* $D > D_\star$, one must not claim that the scalar quadratic stays negative forever. Its second zero is $(125 + \sqrt{1081})/9$. Instead suppose W_4 had all coefficients nonnegative at any $D > D_\star$. Descend to a dimension strictly between $D_\star$ and the smaller of D and this second root. Its scalar coefficient would be nonnegative, whereas the explicit quadratic is negative there. This contradiction proves necessity globally and completes the theorem.

5 Consequence for the finite product

The existing finite family [4, 3] has residues

$$R_{n,k}(x) = \frac{1+x}{2} W_{k-1}(x) \prod_{i=0}^{k-1} \frac{z_i - 1}{z_i - x}, \qquad z_i = 1 + \frac{2(n-i)}{k} > 1, \quad 1 \le k \le n. \tag{19}$$

Each correction has a positive Taylor series, uniformly convergent on $[-1, 1]$ for each fixed pair (n, k). Multiplication by x has nonnegative Gegenbauer matrix elements. The theorem thus controls every $k \neq 3$ for $D \le D_\star$. No uniform angular pole separation in n, k is assumed.

Retain the scalar definition $F_n(D) = \mathbb{E}_D[R_{n,3}(X)]$, with probability density proportional to $(1 - x^2)^{(D-4)/2}$, and the existing strictly decreasing roots d_n of Ref. [3]. New exact rational enclosures give

$$F_{30}(25589/2500) < 0 \quad (\text{approximately } -4.20114224734 \times 10^{-5}), \tag{20}$$

$$F_{29}(127947/12500) > 0 \quad (\text{approximately } 6.60670334173 \times 10^{-6}). \tag{21}$$

The rational dimensions bracket $D_\star$, so $d_{30} < D_\star < d_{29}$. Therefore the analytic cutoff is exactly $n = 30$ for this gap. The finite base becomes $n = 3, \ldots, 29$, requiring 405 non-level-three residues rather than 560. This application uses the prior scalar monotonicity and finite-base results; the present supplement does not claim a fresh replay of the old complete base.

6 Verification and scope

The main checker reconstructs (12) for symbolic D, verifies all 728 rational signs, checks the twenty-term bulk, and checks the shallow formulas. The cutoff checker uses standard-library rational arithmetic and a geometric absolute remainder after Taylor degree 40.

Separately written AI-assisted checks compared 1445 parity-allowed direct Gegenbauer projections with independently expanded Bessel series and every recurrence equation for $m = 0, \ldots, 32$ at $D = 7/2, 10, 51/5, 256/25, 11$. All agreed exactly. A separate cutoff check expanded the three correction series directly, without partial fractions; its rigorous remainder bounds were below 9.46×10^{-48} and 3.39×10^{-47}. These finite checks validate implementations and normalizations; the infinite theorem rests on the analytic reductions and symbolic certificate.

The generating-function and differential-equation arguments are explicitly provided above and inherited from established methods. The old finite-product scalar theorem and base certificates are separate dependencies for the corollary. None of these checks constitutes human expert review, external peer review or formal proof-assistant certification.

A bounded review of the nearest literature found no matching sharp-gap proof. Ordinary type-I positivity up to ten is already established [5]; bosonic positivity up to twenty-six is already proved in Ref. [2]. The recent six-point uniqueness claim [6] is conditional on maximal supersymmetry in four dimensions, scalar parity, a specified completion and further analytic and dispersive assumptions. It neither replaces this certificate nor authorizes a general completion claim for the finite family.

This supplement is best assessed as a strengthening of the existing manuscript. Worldwide priority, publication significance and author approval remain unresolved. No new physical string theory, higher-point factorization, loop unitarity, positive state-space construction or local ultraviolet completion is asserted. The existing public manuscript has not been edited.

References

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[4] L.-Q. Shao and A. Vichi, *Analytic Boundaries of Infinite-Spin-Tower Amplitudes from Hidden Zero*, arXiv:2607.27300v2, 31 August 2026. Section IV.3 and Appendix D.

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