Scalar dominance for the three pole family

AI-assisted research note

5 October 2026

Abstract

For the known planar amplitude with pole spectrum $\{1, 2, r\}$, $3 \le r < 4$, we prove that nonnegativity of the moving-pole scalar partial wave is necessary and sufficient for Gegenbauer positivity at every pole and every spin. The sharp dimension boundary is the unique zero $d(r)$ of an explicit hypergeometric expression. It increases strictly with r and obeys $d(r) = 18/(4-r) - 439/108 + O(4-r)$. The proof combines a positive infinite Taylor tail with an even–odd single-crossing comparison. It does not extrapolate from a spin cutoff. The family and the equidistant endpoint are prior work; novelty of this continuous-parameter extension remains provisional. The result concerns formal tree-level four-point positivity and establishes no physical string completion.

1 The amplitude and the sharp criterion

The finite-product family was constructed and studied by Shao and Vichi [1]. Consider its three-pole member

$$\mathcal{A}_r(s,t) = -\frac{s+t}{st} \prod_{\mu \in \{1,2,r\}} \frac{\mu(\mu - s - t)}{(\mu - s)(\mu - t)}, \qquad r \ge 3. \tag{1}$$

At a massive pole define the positive spectral residue by

$$R_\mu(x) = \lim_{s \to \mu} (\mu - s) \mathcal{A}_r\big(s, \mu(x-1)/2\big), \qquad \lambda = \frac{D-3}{2} > 0. \tag{2}$$

Thus R_μ is the negative of the ordinary complex-analysis residue. Positivity means nonnegative coefficients in its expansion in $C_j^\lambda(x)$, for every integer $j \ge 0$. Let $\mathbb{E}_D$ be normalized integration against $(1-x^2)^{(D-4)/2}\,dx$ on $[-1,1]$, and set $F_r(D) = \mathbb{E}_D[R_r(X)]$.

Theorem 1. *For every $3 \le r < 4$, there is a unique $d(r) > 4$ such that every partial wave at every pole is nonnegative exactly when*

$$3 < D \le d(r), \qquad F_r(d(r)) = 0.$$

At $D = d(r)$ only the moving-pole scalar vanishes among the massive-pole coefficients; all other massive-pole coefficients are strictly positive. The function $d(r)$ is continuous and strictly increasing, and

$$d(r) = \frac{18}{4-r} - \frac{439}{108} + O(4-r) \quad (r \to 4^-). \tag{3}$$

For $r \ge 4$, all partial waves are nonnegative for every $D > 3$.

The bound is sharp implicitly: the unique scalar zero specifies the entire continuous curve. The decimal enclosures below are certified examples, not an explicit elementary formula for that curve.

2 All spins above one are automatic

The moving residue is

$$R_r(x) = \frac{2(x+1)(rx+r-4)(rx+r-2)}{(r-2)(r-1)(3-x)(r+4-rx)(r+2-rx)}. \tag{4}$$

Write

$$A = r^2, \quad B = 2r(r-3), \quad C = (4-r)(r-2), \quad q = \left(\frac{1}{3}, \frac{r}{r+4}, \frac{r}{r+2}\right),$$
$$H(x) = \prod_{i=1}^{3}(1-q_i x)^{-1}, \qquad K = \frac{2}{3(r-2)(r-1)(r+2)(r+4)}. \tag{5}$$

Then $R_r = K(Ax^2+Bx-C)(1+x)H$. For $3 \le r < 4$,

$$A \ge 9, \quad B \ge 0, \quad 0 < C \le 1, \quad S := \sum_i q_i \le \frac{3}{2}. \tag{6}$$

If $(1+x)H(x) = \sum_{m\ge 0} b_m x^m$, then $b_0 = 1$ and $b_m = h_m + h_{m-1}$, where h_m is the complete homogeneous symmetric polynomial in the q_i. Since $h_m \le S h_{m-1}$,

$$b_2 \le S^2 + S \le \frac{15}{4}, \qquad b_m \le S^2 b_{m-2} \quad (m \ge 3). \tag{7}$$

The Taylor coefficients of R_r/K satisfy

$$a_0 = -C, \quad a_1 = B - Cb_1, \quad a_m = Ab_{m-2} + Bb_{m-1} - Cb_m \quad (m \ge 2). \tag{8}$$

Consequently

$$a_2 \ge \frac{21}{4} > 0, \qquad a_m \ge \frac{27}{4} b_{m-2} > 0 \quad (m \ge 3). \tag{9}$$

Every monomial has a nonnegative Gegenbauer expansion for $\lambda > 0$. The only possibly negative Taylor terms have degrees zero and one, so every spin $j \ge 2$ is strictly positive. Each receives a strictly positive contribution from $a_j x^j$. Since $q_i < 2/3$, all series converge absolutely and uniformly on $[-1,1]$, justifying projection. More explicitly, normalizing C_j^λ at $x = 1$ makes the positive expansion weights of x^m sum to one; the total absolute weight is bounded by $\sum_m |a_m|$.

The lower massive poles have the exact factorizations

$$R_1(x) = \frac{2r}{r-1}\frac{1+x}{2r+1-x}\left(1 + \frac{2r-6}{5-x}\right),$$
$$R_2(x) = \frac{r}{r-2}\frac{x(1+x)}{(3-x)(r+1-x)}\left(1 + \frac{r-3}{2-x}\right). \tag{10}$$

Their Taylor coefficients are nonnegative for $r \ge 3$, and they give strictly positive coefficients at every spin. In R_2 the scalar receives a positive contribution from x^2. The massless spectral residue is the constant one.

3 A scalar comparison that controls spin one

For $x \in [0,1]$, put $t = x^2$ and

$$H_e = \frac{H(x)+H(-x)}{2}, \qquad h = \frac{H(x)-H(-x)}{H(x)+H(-x)} = \tanh\left(\sum_i \operatorname{artanh}(q_i x)\right),$$
$$z = xh, \qquad W = H_e(1+z), \qquad V = H_e(1+h/x), \qquad k(t) = \frac{tV}{W} = \frac{t+z}{1+z}. \tag{11}$$

At zero these expressions mean their continuous limits. We have $W > 0$, $0 \le h < 1$, and z is increasing. Hence

$$k'(t) = \frac{1+z+(1-t)z'}{(1+z)^2} > 0, \qquad 0 \le k(t) \le \sqrt{t}, \tag{12}$$

with strict second inequality for $0 < t < 1$. The latter is equivalent to $x(1-x)(h-1) \le 0$.

The even part of R_r/K is $Wp(t)$, and the even part of xR_r/K is

$$W\big[k(t)p(t) + B(t-k(t)^2)\big], \qquad p(t) = At - C + Bk(t). \tag{13}$$

Because p is strictly increasing, $p(0) = -C < 0$, and $p(1) = A + B - C > 0$, it has exactly one zero $t_0 \in (0,1)$. Thus

$$[k(t) - k(t_0)]p(t) \ge 0. \tag{14}$$

After integration against W times the pushed-forward Gegenbauer measure, the inequality is strict: the density is positive in $(0,1)$, and the integrand is positive away from t_0. Equations (12)–(13) therefore give

$$\mathbb{E}_D[R_r] \ge 0 \quad \Longrightarrow \quad \mathbb{E}_D[xR_r] > 0. \tag{15}$$

Since $C_1^\lambda(x) = 2\lambda x$, this is precisely scalar dominance over spin one. The same comparison works for any positive continuous symmetric measure; degenerate measures need not give strict positivity.

4 The unique zero and its monotonicity

The normalized moments are

$$M_n(D) = \mathbb{E}_D[X^{2n}] = \frac{(1/2)_n}{((D-1)/2)_n}, \qquad \frac{F_r(D)}{K} = -C + \sum_{n\ge 1} a_{2n} M_n(D). \tag{16}$$

By (9), this scalar is continuous and strictly decreasing in $D > 3$, and tends to $-KC < 0$. Moreover

$$F_r(4)/K \ge -1 + (21/4)/3 = 3/4 > 0. \tag{17}$$

This proves existence and uniqueness of $d(r)$; the earlier sections establish necessity, sufficiency, and endpoint strictness. Continuity of the unique zero follows from continuity in (r, D) and its strict sign crossing.

To prove monotonicity in r, let e_i be the elementary symmetric polynomials in q. Direct evenization gives

$$W_r(t) = \frac{1 + (e_1+e_2)t + e_3 t^2}{\prod_i (1-q_i^2 t)}. \tag{18}$$

For $r_2 > r_1$, all q_i are nondecreasing and two strictly increase. The ratios of the numerator coefficients, ordered by degree, are nondecreasing: 1, the ratio of $e_1 + e_2$, and the ratio of e_3. Indeed

the last is the product of all individual q_i ratios and bounds the ratio of each monomial in $e_1 + e_2$. Nondecreasing coefficient ratios imply a nondecreasing quotient of positive polynomials, by pairing terms in $N_2'N_1 - N_2N_1'$. Each factor $(1 - q_i(r_1)^2t)/(1 - q_i(r_2)^2t)$ is also nondecreasing. Hence W_{r_2}/W_{r_1} increases in t.

Next write $\bar{p} = p/C = \alpha t - 1 + \beta k$, with $\alpha = A/C$, $\beta = B/C$. Both increase strictly because

$$\alpha' = \frac{2r(3r-8)}{C^2} > 0, \qquad \beta' = \frac{2(3r^2 - 16r + 24)}{C^2} > 0. \tag{19}$$

Also k is nondecreasing in r at fixed t, so $\bar{p}_{r_2}(t) > \bar{p}_{r_1}(t)$ for $t > 0$. At $D = d(r_1)$, $\mathbb{E}_D[W_{r_1}\bar{p}_{r_1}] = 0$. Multiplication of this single-crossing integrand by the increasing ratio W_{r_2}/W_{r_1} cannot make its integral negative. Replacing $\bar{p}_{r_1}$ by $\bar{p}_{r_2}$ then makes it strictly positive. Thus $F_{r_2}(d(r_1)) > 0$ and $d(r_2) > d(r_1)$.

5 The endpoint and the exact boundary formula

For $r \geq 4$, $C \leq 0$ and $B \geq 0$; the entire moving residue has nonnegative Taylor coefficients. Its scalar at $r = 4$ is strictly positive for every finite $D > 3$. Continuity and monotonicity force $d(r) \to \infty$ as $r \to 4^-$. Set $\epsilon = 4 - r$. Exact coefficient expansions give

$$C = 2\epsilon - \epsilon^2, \quad a_2 = 36 - 40\epsilon + O(\epsilon^2), \quad a_4 = 641/9 + O(\epsilon). \tag{20}$$

The coefficients are uniformly absolutely summable near $r = 4$. Since $M_n(D) \leq M_3(D) = O(D^{-3})$ for $n \geq 3$, the scalar equation is uniformly

$$\begin{aligned} 0 = & -2\epsilon + \epsilon^2 + \frac{36 - 40\epsilon + O(\epsilon^2)}{D-1} \\ & + \frac{641/3 + O(\epsilon)}{(D-1)(D+1)} + O(D^{-3}). \end{aligned} \tag{21}$$

First $\epsilon d(r) \to 18$, and then $d(r) = 18/\epsilon + O(1)$. Substitution of $D = 18/\epsilon + c$ leaves coefficient $-c/9 - 439/972$ at order ϵ^2, proving (3) with a uniform $O(\epsilon)$ remainder.

For an explicit implicit specification, set $G_D(q) = {}_2F_1(1, 1/2; (D-1)/2; q^2)$. Partial fractions give

$$\begin{aligned} F_r(D) = & -\frac{2}{(r-2)(r-1)} + \frac{4r(r+1)}{(r-2)(r-1)(r+2)} G_D\left(\frac{r}{r+2}\right) \\ & - \frac{4r(r+1)(r+2)}{(r-2)^2(r-1)(r+4)} G_D\left(\frac{r}{r+4}\right) + \frac{16(2r-1)}{3(r-2)^2(r-1)} G_D(1/3). \end{aligned} \tag{22}$$

The unique zero of (22) is the exact sharp boundary.

6 Certificates and research status

The accompanying Python source checks the residue, even–odd, partial-fraction, and asymptotic identities symbolically. All displayed root enclosures are certified with exact rational arithmetic. For $G_D(q) = \sum_{n \geq 0} q^{2n} M_n(D)$, truncation at N has the rigorous remainder

$$0 \leq \text{tail} \leq \frac{M_{N+1}(D) q^{2N+2}}{1 - q^2}. \tag{23}$$

Using $N = 80$ in the three signed terms of (22) proves opposite strict signs at these rational endpoints:

r	Lower bound on $d(r)$	Upper bound on $d(r)$
3	13.1598342617	13.1598342618
13/4	19.4064298137	19.4064298138
7/2	31.6112099674	31.6112099675
15/4	67.7852008807	67.7852008808
39/10	175.8777416300	175.8777416301
399/100	1795.9295838288	1795.9295838289

Run `python exact_checks.py` with Python and SymPy. It regenerates the exact scalar intervals in `exact_certificate.json`. The numerical certificates illustrate the curve; the elementary proof controls the infinite set of spins and the full real parameter interval.

Shao and Vichi [1], Section IV.2, give the larger moving-endpoint family and prove its four-dimensional positivity; Section IV.3 studies critical dimensions for equidistant poles. The existing finite-product manuscript [2] already establishes scalar dominance at $r = 3$. A bounded comparison with the nearest cited literature did not locate the present continuous-r criterion. This does not establish worldwide priority; a specialist should assess novelty and significance.

AI disclosure and limits This note's derivation, code, and separate analytic checks were produced with OpenAI assistance. The identities and rational signs were executed, and the all-spin proof received a separately generated mathematical review. This is not human peer review or proof-assistant certification. No higher-point factorization, local finite-spin interpretation, loop unitarity, positive state-space construction, or physical string theory follows from the theorem. No public submission or publication has been made.

References

[1] L.-Q. Shao and A. Vichi, *Analytic Boundaries of Infinite-Spin-Tower Amplitudes from Hidden Zero*, arXiv:2607.27300v2, 31 August 2026. https://arxiv.org/html/2607.27300v2.

[2] Y. Pan, *Sharp all spin dimension bounds for finite Veneziano products*, manuscript snapshot, 3 October 2026. Fixed repository source snapshot.